Definition 1. Algorithms which change the boundary of the solution region in order to find the optimal solution of an integer programme are called *cut algorithms*.

The branch-and-bound algorithm does this by splitting the solution region into two and then discard the one which does not contain the optimal solution.

The Gomory algorithm, on the other hand, reduce the feasible region with the help of a new constraint without the region being splitted.

Definition 2. We call *branching* a process by which a programme whose solution contains a non-integral

$$j < x_i < k$$

is made into two separate programmes having the additional constraint

$$x_i \leq j$$

in one, and

$$x_i \geq k$$

in the other, the objective together with all the constraints of the original problem of which remain the same. Here j and k are positive integers and $j < k$.

Definition 3. In the branch-and-bound algorithm, if the objective is maximisation, the value of the objective obtained when the first integral approximation occurs is said to be the lower bound for the problem, and if the objective is minimisation it is said to be the upper bound of the same.

Algorithm 1 *Branch-and-bound algorithm for integer programming.*

find first approximation
while approximations not all integers **do**
  choose x_i from all non-integral variables such that
    $\min\left(\left|x_i - \lfloor x_i \rfloor\right|, \left|x_i - \lceil x_i \rceil\right|\right)$ is maximised
  branch
  choose the branch whose value of the objective
    is maximum
endwhile
solution←last approximation

Example 1. (*Problem 6.9; Bronson, 1982*)

$$\begin{aligned} \text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\ \text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

Solve by branch-and-bound algorithm.

Draw a simplex table of Programme 1.

		x_1	x_2	x_3	x_4	
		1	2	1	0	
x_4	0	2	3	3	1	11
		-1	-2	-1	0	0

Replace x_4 for x_2 as the basic variable.

	x_1	x_2	x_3	x_4	
x_2	$\frac{2}{3}$	1	1	$\frac{1}{3}$	$\frac{11}{3}$
	$\frac{1}{3}$	0	1	$\frac{2}{3}$	$\frac{22}{3}$

$$x_2^* = \frac{11}{3} = 3.6,\ x_1^* = x_3^* = x_4^* = 0,\ z^* = \frac{22}{3}$$

Since $3 < x_2^* < 4$, branch into two programmes, namely Programme 1 where $x_2 \leq 3$, and Programme 2 where $x_2 \geq 4$. Consider first Programme 2.

$$
\begin{aligned}
\text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\
\text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\
& x_2 \leq 3 \\
\text{with:} \quad & \text{all variables non-negative and integral}
\end{aligned}
$$

Use the simplex method in a tabulated form.

		x_1	x_2	x_3	x_4	x_5	
		1	2	1	0	0	
x_4	0	2	3	3	1	0	11
x_5	0	0	1	0	0	1	3
		-1	-2	-1	0	0	0

Replace the basic variable x_5 with x_2.

	x_1	x_2	x_3	x_4	x_5	
x_4	2	0	3	1	-3	2
x_2	0	1	0	0	1	3
	-1	0	-1	0	2	6

Replace the basic variable x_4 with x_1.

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	1
x_2	0	1	0	0	1	3
	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	7

$x_1^* = 1,\ x_2^* = 3,\ x_3^* = x_4^* = x_5^* = 0,\ z^* = 7$

Then consider Programme 3.

$$
\begin{aligned}
\text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\
\text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\
& x_2 \geq 4 \\
\text{with:} \quad & \text{all variables non-negative and integral}
\end{aligned}
$$

Draw a table for the two-phase method.

		x_1	x_2	x_3	x_4	x_5	x_6	
		1	2	1	0	0	$-M$	
x_4	0	2	3	3	1	0	0	11
x_6	$-M$	0	1	0	0	-1	1	4
		-1	-2	-1	0	0	0	0
		0	-1	0	0	1	-1	-15

Change x_4 for x_2 in the basic variables.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	$\frac{2}{3}$	1	1	$\frac{1}{3}$	0	0	$\frac{11}{3}$
x_6	$-\frac{2}{3}$	0	-1	$-\frac{1}{3}$	-1	1	$\frac{1}{3}$
	$\frac{1}{3}$	0	1	$\frac{2}{3}$	0	0	$\frac{22}{3}$
	$\frac{2}{3}$	0	1	$\frac{1}{3}$	1	-1	$-\frac{34}{3}$

The coefficient parts of the row corresponding to the basic variable x_6 and the last row cancel each other. The optimal result is $x_2^* = \frac{11}{3}$, $x_1^* = x_3^* = x_4^* = x_5^* = x_6^* = 0$ and $z^* = \frac{22}{3}$.

$$(3)\ x_2 \geq 4,\ z^* = \frac{22}{3},\ \left(0, \frac{11}{3}\right)$$

$$(1)\ z^* = \frac{22}{3},\ \left(0, \frac{11}{3}\right)$$

$$(2)\ x_2 \leq 3,\ z^* = 7,\ (1, 3)$$

Therefore the solution is $x_1^* = 1$, $x_2^* = 3$, $x_3^* = x_4^* = x_5^* = 0$, and $z^* = 7$.

#

Algorithm 2 *Gomory algorithm for integer programming.*

while solution not wholly all integers **do**
  choose one non-integral optimal approximation
  write a relation from the row where that variable is basic
  rewrite the relation to make all fractional coefficients
    some integer plus a proper fraction
  move all the fractions to LHS, and all the non-fractions
    to RHS
  write a new constraint as LHS ≥ 0
  find the solution for the original problem together
    with the new constraint
endwhile

Example 2. (*Problem 7.1; Bronson, 1982*)

$$\begin{aligned} \text{maximise:} \quad & z = 2x_1 + x_2 \\ \text{subject to:} \quad & 2x_1 + 5x_2 \leq 17 \\ & 3x_1 + 2x_2 \leq 10 \\ \text{with:} \quad & x_1, x_2 \, \text{non-negative and integral} \end{aligned}$$

Use cut algorithm.

Solve

Find the first approximation of Programme 1 normally using the simplex method.

		x_1	x_2	x_3	x_4	
		2	1	0	0	
x_3	0	2	5	1	0	17
x_4	0	3	2	0	1	10
		-2	-1	0	0	0

Since $\frac{10}{3} < \frac{17}{2}$, we know that 3 is the pivot element, and therefore we replace the basic variable x_4 with x_1.

	x_1	x_2	x_3	x_4	
x_3	0	$\frac{11}{3}$	1	$-\frac{2}{3}$	$\frac{31}{3}$
x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$
	0	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{20}{3}$

We have $x_1^* = \frac{10}{3}$, $x_3^* = \frac{31}{3}$, $x_2^* = x_4^* = 0$ and $z^* = \frac{20}{3}$.

Since both x_1^* and x_3^* are non-integers, arbitrarily choose the former to generate a new constraint. Then our Programme 2 becomes,

$$\begin{aligned} x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_4 &= \frac{10}{3} = 3 + \frac{1}{3} \\ \frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} &= 3 - x_1 \\ \frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} &\geq 0 \\ \frac{2}{3}x_2 + \frac{1}{3}x_4 &\geq \frac{1}{3} \\ 2x_2 + x_4 &\geq 1 \end{aligned}$$

and our new programme becomes

$$
\begin{array}{rl}
\text{maximise:} & z = 2x_1 + x_2 + 0x_3 + 0x_4 \\
\text{subject to:} & \dfrac{11}{3}x_2 - \dfrac{2}{3}x_4 = \dfrac{31}{3} \\
& x_1 + \dfrac{2}{3}x_2 - \dfrac{1}{3}x_4 = \dfrac{10}{3} \\
& 2x_2 + x_4 \geq 1 \\
\text{with:} & \text{all variables non-negative and integral}
\end{array}
$$

		x_1	x_2	x_3	x_4	x_5	x_6	
		2	1	0	0	0	$-M$	
x_1	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	0	$\frac{10}{3}$
x_3	0	0	$\frac{11}{3}$	1	$-\frac{2}{3}$	0	0	$\frac{31}{3}$
x_6	$-M$	0	2	0	d1	-1	1	1
		-2	-1	0	0	0	0	0
		0	-2	0	-1	1	-1	-1

Now x_2 replaces x_6 in the basic variables and becomes the pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	3
x_3	0	0	1	$-\frac{15}{6}$	$\frac{11}{6}$	$-\frac{11}{6}$	$\frac{17}{2}$
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	-2	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	0	0	0	0

This becomes,

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	0	$\frac{1}{3}$	3
x_3	0	0	1	$-\frac{5}{2}$	$\frac{11}{6}$	$\frac{17}{2}$
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{13}{2}$

Then our first approximation of Programme 2 is $x_1^* = 3$, $x_2^* = \frac{1}{2}$, $x_3^* = \frac{17}{2}$, $x_4^* = x_5^* = 0$, and $z^* = \frac{13}{2}$.

Arbitrarily choose x_2^* to generate the new constraint.

$$
\begin{aligned}
x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 &= \frac{1}{2} \\
\frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &= -x_2 \\
\frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &\geq 1 \\
x_4 - x_5 &\geq 1
\end{aligned}
$$

Then our Programme 3 becomes,

$$
\begin{aligned}
\text{maximise:} \quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\
\text{subject to:} \quad & x_1 + \frac{1}{3}x_5 = 3 \\
& x_3 - \frac{5}{2}x_4 + \frac{11}{6}x_5 = \frac{17}{2} \\
& x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 = \frac{1}{2} \\
& x_4 - x_5 \geq 1 \\
\text{with:} \quad & \text{all variables non-negative and integral}
\end{aligned}
$$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
		2	1	0	0	0	0	$-M$	
x_1	0	1	0	0	0	$\frac{1}{3}$	0	0	3
x_2	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
x_3	0	1	0	0	$-\frac{5}{2}$	$\frac{11}{6}$	0	0	$\frac{17}{2}$
x_7	$-M$	0	0	0	1	-1	-1	1	1
		-2	-1	0	0	0	0	0	0
		0	0	0	-1	1	1	-1	-1

Then x_4 replaces the basic x_7 to become the pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	1	0	0	0	$-\frac{2}{3}$	$-\frac{5}{2}$	11
x_4	0	0	0	1	-1	-1	1
	-2	-1	0	0	0	0	0

Next, x_1 remains basic and becomes a pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	0	0	0	0	-1	$-\frac{5}{2}$	8
x_4	0	0	0	1	-1	-1	1
	0	-1	0	0	$\frac{1}{3}$	0	6

This becomes

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	0	0	0	0	-1	$-\frac{5}{2}$	8
x_4	0	0	0	1	-1	-1	1
	0	0	0	0	$\frac{1}{3}$	$\frac{1}{2}$	6

The optimum point for Programme 3 is then, $x_1^* = 3$, $x_3^* = 8$, $x_4^* = 1$, $x_2^* = x_5^* = x_6^* = 0$ and $z^* = 6$.

Therefore the solution to the original problem Programme 1 is $x_1^* = 3$, $x_2^* = 0$ at the objective value $z^* = 6$.

#

Definition 4. A *transportation problem* involves m *sources* each of which supplies a_i, $i = 1, \ldots, m$, units of a certain product, and n *destinations* each of which requires b_i, $i = 1, \ldots, n$, units of the same. The problem may be stated as following.

$$\text{maximise:} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to:} \quad \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n$$

with: all x_{ij} non-negative and integral

The total supply and the total demand are assumed to be equal. Were this not so, a fictitious destination or a fictitious source is added.

Definition 5. The *north-west corner rule* finds an initial basic solution for the transportation algorithm of the integer programming. It begins with the (1,1) cell in the $m \times n$ table, and allocates as many units as possible to x_{11} violating neither the constraints of supply, that is the summation along each row, nor those of demand, that is the summation along each column. Then carry on moving for each step either right or downwards, until we reach the lower-right corner, x_{mn}.

Definition 6. A *loop*, which is a sequence of cells in the table used for finding the solution in the transportation problem, has the following properties.

a. each pair of consecutive cells is on either the same row or the same column
b. no three, or in fact any odd-numbered, consecutive cells lie in the same row or column
c. the first and the last cells are on the same row or column
d. the path along the loop is self-avoiding, that is no cells appear more than once in the sequence

Algorithm 3 *Transportation algorithm.*

while optimal solution not attained **do**

  find an initial, basic feasible solution using, for instance, the North-west corner rule

  let *either* $u_i = 0$ *or* $v_j = 0$ *depending on whether* the i^{th}-row or the j^{th}-column has the maximum number of basic solutions

  find all u_i and v_j, $i = 1, \ldots, m$ and $j = 1, \ldots, n$ from $u_i + v_j = c_{ij}$ for basic variables, and from $c_{ij} - u_i - v_j$ for non-basic variables

  improve the solution

endwhile

Note 1. In a transportation problem, optimal solution is achieved when

$$c_{ij} - u_i - u_j \geq 0$$

for all transportation costs per unit c_{ij} of all non-basic variables.